MULTIPLIERS OF TRIGONOMETRIC SERIES AND POINTWISE CONVERGENCE(1)

BY

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Introduction. In a recent paper M. Weiss and A. Zygmund [7] have studied the pointwise convergence of a trigonometric series $\sum a_n e^{inx}$ when the multipliers $\lambda_n = |n|^{i\gamma}$ (γ real) are applied to it. The proof of their result makes use of Peano derivatives in L^p, which bear a close connection with the t_u^p classes of A. P. Calderón and A. Zygmund [1]. In this paper we prove that conditions of Marcinkiewicz type for a multiplier are enough to preserve the t_u^p classes (Theorems 1 and 2). As a consequence we obtain results on pointwise convergence for multipliers which satisfy a variational condition of Marcinkiewicz type (Theorem 3).

I. **Notation.** All functions to be considered in this paper are periodic with period 2π. We define

$$\|f\|_p = \left(\int_{-\pi}^{\pi} |f(x)|^p\,dx\right)^{1/p} \quad \text{and} \quad \mathscr{L}^p = \{f;\ \|f\|_p < \infty\}.$$

Definition 1. Let $u \geqq 0$, by $T_u^p(x_0)$, $1 \leqq p < \infty$; we denote the class of functions f, belonging to $\mathscr{L}^p$, and such that there exists a polynomial $P_m(x)$ of degree m, $m < u$ ($P_m = 0$ if $u = 0$), so that

$$\left(\frac{1}{h}\int_{-h}^{h} |f(x-x_0) - P_m(x)|^p\,dx\right)^{1/p} \leqq Ah^u, \tag{1.1}$$

for $0 < h \leqq \pi$, with A independent of h. If $P_m(x) = \sum a_n x^n$, we write $T_u^p(x_0, f) = \|f\|_p + \sum |a_n| + \inf\{A\}$.

Definition 2. Let $f \in T_u^p(x_0)$ we shall say that $f \in t_u^p(x_0)$ if and only if there exists a polynomial $P_m(x)$ of degree m, $m \leqq u$, such that

$$\left(\frac{1}{h}\int_{-h}^{h} |f(x-x_0) - P_m(x)|^p\,dx\right)^{1/p} = o(h^u). \tag{1.2}$$

C^∞ will denote the class of infinitely differentiable functions.

II. **Multipliers preserving $T_u^p(x_0)$ and $t_u^p(x_0)$.** We start by stating some properties of the spaces $T_u^p(x_0)$ and $t_u^p(x_0)$ (see [1]):

(1) $T_u^p(x_0)$ is a Banach space with the norm $T_u^p(x_0, \cdot)$.

(2) $t_u^p(x_0)$ is a closed subspace of $T_u^p(x_0)$; C^∞ is dense in $t_u^p(x_0)$.

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THEOREM 1. *Let $k(x) \in \mathscr{L}^1$, such that*:

(1) $|(d/dx)^j k(x)| \leqq C/|x|^{j+1}$ *for* $0 \leqq j \leqq r$.

(2) *If* $K(f) = (1/\pi) \int_{-\pi}^{\pi} k(x-y)f(y)\, dy$, *then* $\|Kf\|_p \leqq C\|f\|_p$.

Then K is a continuous operator from $T_u^p(x_0)$ *to* $T_u^p(x_0)$, *and from* $t_u^p(x_0)$ *to* $t_u^p(x_0)$, *for* $u \leqq r$. *Moreover* $T_u^p(x_0; Kf) \leqq B_u C T_u^p(x_0, f)$, *where* B_u *is a constant depending on u only.*

Proof. The proof is similar to that of Lemma 5.1 in [1].

Without loss of generality we may assume that $x_0 = 0$. We will show first the preservation of $T_u^p(x_0)$.

Let $P_m(x)$ be the polynomial of (1.1) for $f(x)$.

Take $\phi(x) \in C^\infty$ such that: $\phi(x) = 1$ for $|x| < \pi/4$ and $\phi(x) = 0$ for $|x| > \pi/2$. Set $f(x) = f_1(x) + f_2(x)$ where $f_2(x) = P_m(x)\phi(x)$. Since for $h \leqq \pi/4$,

$$\int_{-h}^{h} |f_2(x) - P_m(x)|^p\, dx = 0,$$

then it is clear that $T_u^p(0, f_2) \leqq B_u T_u^p(0, f)$.

On the other hand, if $\psi(x) \in C^\infty$,

$$K(\psi)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} k(y)[\psi(x-y) - \psi(x)]\, dy + \frac{\psi(x)}{\pi} \int_{-\pi}^{\pi} k(y)\, dy.$$

Using then the fact that $|k(y)| \leqq C/|y|$ and that $\left|\int_{-\pi}^{\pi} k(y)\, dy\right| \leqq C$ (which follows from condition (2)); it follows that $|K(\psi)(x)| \leqq B_\psi C$. Similarly $|(d/dx)^j K(\psi)(x)| \leqq B_\psi C$. Hence $T_u^p(0, K(\psi)) \leqq B_\psi C$. Applying this observation to $x^n \phi(x)$:

$$T_u^p(0, K(f_2)) \leqq \sum_{n<u} |a_n| T_u^p(0, K(x^n\phi)) \leqq B_u C T_u^p(0, f). \tag{1.3}$$

We pass now to consider $f_1(x)$. Clearly

$$\left(\frac{1}{h} \int_{-h}^{h} |f_1(x)|^p\, dx\right)^{1/p} \leqq T_n^p(0, f) h^u, \quad \text{for } h \leqq \frac{\pi}{4}.$$

CLAIMS:

$$\int_{-h}^{h} |f_1(x)|\, |x|^{-j}\, dx \leqq B_u T_u^p(0, f) h^{u+1-j}, \quad \text{for } 1 \leqq j < u+1. \tag{1.4}$$

$$\int_{\pi \geqq |x| \geqq h} |f_1(x)|\, |x|^{-j}\, dx \leqq B_u T_u^p(0, f) h^{u+i-j}, \quad \text{for } u+1 \leqq j. \tag{1.5}$$

We postpone the proof of (1.4) and (1.5) and proceed to show that $T_u^p(0, K(f_1)) \leqq B_u C T_u^p(0, f)$. Expanding $k(x)$ by Taylor's formula, we have

$$
\begin{aligned}
K(f_1) = {} & \frac{1}{\pi}\int_{-h}^{h} k(x-y)f_1(y)\,dy+\sum_{n<u}\frac{x^n}{n!}\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{d}{dy}\right)^n k(-y)f_1(y)\,dy \\
& +\frac{x^l}{l!}\frac{1}{\pi}\int_{h\leqq|x|\leqq\pi}\left(\frac{d}{dy}\right)^l k(\theta x-y)f_1(y)\,dy \\
& -\sum_{n<u}\frac{x^n}{n!}\frac{1}{\pi}\int_{-h}^{h}\left(\frac{d}{dy}\right)^n k(-y)f_1(y)\,dy
\end{aligned}
\tag{1.6}
$$

where $u\leqq l<u+1$. Set $b_n=(1/n!\pi)\int_{-\pi}^{\pi}(d/dy)^n k(-y)f_1(y)\,dy$; then using condition (1) and (1.4):

$$
|b_n| \leqq B_u\int_{-\pi}^{\pi}|f_1(y)|\,|y|^{-n-1}\,dy \leqq B_u C T_u^p(0,f) \qquad (0\leqq n<u). \tag{1.7}
$$

Moreover

$$
\int_{-h/2}^{h/2}\left|\int_{-h}^{h}k(x-y)f_1(y)\,dy\right|^p dx \leqq C^p\int_{-h}^{h}|f_1(y)|^p\,dy \leqq C^pT_n^p(0,f)h^{pu+1} \tag{1.8}
$$

and for $n<u$

$$
\begin{aligned}
& \int_{-h/2}^{h/2}\left|\frac{x^n}{n!\pi}\int_{-h}^{h}\left(\frac{d}{dy}\right)^n k(-y)f_1(y)\,dy\right|^p dx \\
& \qquad \leqq B_uC^p\int_{-h/2}^{h/2}|x|^{np}\left(\int_{-h}^{h}|f_1(y)|\,|y|^{-n-1}\,dy\right)^p dx \\
& \qquad \leqq B_uC^pT_u^p(0,f)h^{up+1}.
\end{aligned}
\tag{1.9}
$$

Finally using (1.5)

$$
\begin{aligned}
& \int_{-h/2}^{h/2}\left|\frac{x^l}{l!}\int_{\pi\geqq|x|>h}\left(\frac{d}{dy}\right)^l K(\theta x-y)f_1(y)\,dy\right|^p dx \\
& \qquad \leqq B_uC^p\int_{-h/2}^{h/2}|x|^{lp}\left(\int_{\pi\geqq|x|>h}|f_1(y)|\,|y|^{-l-1}\,dy\right)^p dx \\
& \qquad \leqq B_uC^pT_u^p(0,f)h^{up+1}.
\end{aligned}
\tag{1.10}
$$

From (1.6), (1.7), (1.8), (1.9) and (1.10) it follows that $T_u^p(0,K(f_1))\leqq B_uCT_u^p(0,f)$, this inequality together with (1.3) proves the first part of the theorem.

For the second part it is enough to observe that if $f\in C^\infty$, then $K(f)\in C^\infty$, and C^∞ is dense in $t_u^p(x_0)$.

Proof of Claims (1.4), (1.5). Assume that $(\int_{-h}^{h}|f(x)|^p\,dx)^{1/p}\leqq Ah^{u+1/p}$; set $g(t)=\int_{-t}^{t}|f(x)|\,dx\leqq At^{u+1}$. Then for $0\leqq j<r-1$:

$$
\begin{aligned}
\int_{-h}^{h}|f(x)|\,|x|^{-j}\,dx &= \lim_{\varepsilon\to 0}\int_{\varepsilon}^{h}t^{-j}\,d(g(t)) = \lim_{\varepsilon\to 0}\left(t^{-j}g(t)|_{\varepsilon}^{h}+j\int_{\varepsilon}^{h}g(t)t^{-j-1}\,dt\right) \\
&\leqq Ah^{u+1-j}+A\int_0^h t^{u-j}\,dt \leqq B_uAh^{u+1-j},
\end{aligned}
$$

and (1.4) follows. (1.5) can be proved using a similar argument.

We shall discuss next what conditions on the Fourier coefficients of $k(x)$ guarantee properties (1) and (2).

LEMMA. *Let* $\{\lambda_n\}_{n=-\infty}^{+\infty}$, *such that*

(a) $\lambda_n=0$ *for* $|n|>N$,

(b) $|\lambda_n|<C$ *and* $\sum_{\pm 2^k}^{\pm 2^{k+1}} |n|^r|\Delta^{r+1}(\lambda_n)|\leqq C,\ r\geqq 1\ (k=0,1,2,\dots)$

$$\Delta\lambda_n = \lambda_n-\lambda_{n+1}; \qquad \Delta^r\lambda_n = \Delta(\Delta^{r-1}\lambda_n).$$

Then $k(x)=\sum_{-\infty}^{\infty} \lambda_n e^{inx}$ *satisfies property* (1) *for* $j\leqq r-1$, *and property* (2).

Proof. Condition (b) implies

$$|\Delta^k\lambda_n| \leqq BC|n|^{-k} \quad \text{when } k \leqq r. \tag{1.11}$$

Property (2) is now a consequence of the Marcinkiewicz multiplier theorem (see [3], [9, II, p. 232]) since

$$\sum_{\pm 2^k}^{\pm 2^{k+1}} |\Delta\lambda_n| \leqq BC \sum_{\pm 2^k}^{\pm 2^{k+1}} \frac{1}{|n|} \leqq BC$$

and therefore $\|K(f)\|_p\leqq B_pC\|f\|_p$ where B_p depends on p only. To prove property (b), set

$$Z_n^{[k]}(x) = \frac{e^{i(n+k)x}}{(e^{ix}-1)^k};$$

observe that

$$Z_n^{[k]}(x)-Z_{n+1}^{[k]}(x) = Z_{n+1}^{[k-1]}(x), \tag{1.12}$$

and

$$|Z_n^{[k]}(x)| \leqq 1/|x|^k. \tag{1.13}$$

Set, for $x\neq 0$, $m=[1/|x|]$ the integer part of $1/|x|$, then, using (1.12) and summation by parts,

$$\begin{aligned}\left(\frac{d}{dx}\right)^j k(x) &= \sum_{n=-N}^{\infty} (in)^j\lambda_n e^{inx} = \sum_{n=-N}^{\infty} \Delta^{j+2}(\lambda_n(in)^j)Z_n^{[j+2]}(x)\\ &= \sum_{|n|\leqq M} \Delta^{j+2}(\lambda_n(in)^j)Z^{[j+2]}(x)+\sum_{|n|>M} \Delta^{j+2}(\lambda_n(in)^j)Z_n^{[j+2]}(x)\\ &= P+Q.\end{aligned}$$

To estimate Q, let $2^s\leqq M\leqq 2^{s+1}$. Using the estimates (1.11), (1.13) and condition (b):

$$\begin{aligned}|Q| &\leqq \frac{B}{|x|^{j+2}}\sum_{k=s}^{\infty}\left\{\sum_{n=\pm 2^k}^{\pm 2^{k+1}}\left(\sum_{l=0}^{j+2}|\Delta^{j+2-l}(\lambda_n)|(|n|^{j-l})\right)\right\}\\ &\leqq \frac{BC}{|x|^{j+2}}\sum_{k=s}^{\infty} 2^{-k} \leqq \frac{BC}{|x|^{j+1}}.\end{aligned}$$

To estimate P, use a summation by parts argument and (1.12):

$$\begin{aligned} P = \sum_{n=-M}^{M+2} \Delta^j(\lambda_n(in)^j)Z_n^{[j]}(x) - \{[Z_{M+2}^{[j+1]}(x) - Z_{-M}^{[j+1]}(x)]\, \Delta^{j+1}(\lambda_{M+2}(i(M+2))^j) \\ + Z_{-M}^{[j+1]}(x)[\Delta^j(\lambda_{-M}(iM)^j) - \Delta^j(\lambda_{M+1}(i(M+1))^j)]\} \\ + \{[Z_{M+1}^{[j+2]}(x) - Z_{-M}^{[j+2]}(x)]\Delta^{j+1}(\lambda_{M+1}(i(M+1))^j) \\ + Z_{-M}^{[j+2]}(x)[\Delta^{j+2}(\lambda_{-M}(-M)^j) - \Delta^{j+2}(\lambda_M(iM)^j)]\}. \end{aligned}$$

Hence using (1.11) and (1.13)

$$|P| \leqq BC\frac{M}{|x|^j} + \frac{BC}{|x|^{j+1}} + \frac{BC}{M|x|^{j+2}} \leqq \frac{BC}{|x|^{j+1}}.$$

The lemma follows. As a consequence of the lemma we have

THEOREM 2. *Let* $\{\lambda_n\}_{n=-\infty}^{\infty}$ *such that* $|\lambda_n| \leqq C$ *and*

$$\sum_{\pm 2^k}^{\pm 2^{k+1}} |\Delta^{r+1}(\lambda_n)|\; |n|^r \leqq C, \quad r \geqq 1 \qquad (k = 0, 1, 2, \dots).$$

Define for $f \in C^\infty$, $f(x) = \sum a_n e^{inx}$, $\bigwedge(f) = \sum_{n=-\infty}^{\infty} \lambda_n a_n e^{inx}$. *Then, for* $u \leqq r-1$ $T_u^p(x_0, \bigwedge f) \leqq B_{u,p} C T_u^p(x_0, f)$, *and as a consequence* $\bigwedge$ *can be extended to be a continuous mapping from* $t_u^p(x_0)$ *into* $t_u^p(x_0)$. (*Since* C^∞ *is dense in* $t_u^p(x_0)$.)

Proof. Let $\phi(t) \in C^\infty(-\infty, \infty)$, such that $\phi(t) = 1$ for $|t| \leqq 1$ and $\phi(t) = 0$ for $|t| \geqq 2$. Set $\mu_n = \lambda_n \phi(n/N)$ where N is a positive integer. Then, for $f \in C_\infty$ as before,

$$\bigwedge\nolimits_N(f) = \sum \mu_n a_n e^{inx} = \frac{1}{\pi}\int_{-\pi}^{\pi} k_N(x-y) f(y)\, dy,$$

where $k_N(x) = \sum_{n=-2N}^{2N} \mu_n e^{inx}$. The theorem becomes an immediate consequence of Theorem 1 and the lemma once we observe that:

(i) Since $|(d/dx)^i \phi(x/N)| \leqq B/N$ for $|x| \leqq 2N$ and it vanishes for $|x| \geqq 2N$, then

$$|\mu_n| \leqq BC \quad \text{and} \quad \sum_{\pm 2^k}^{\pm 2^{k+1}} |\Delta^{r+1}(\mu_n)|\; |n|^r \leqq BC.$$

(ii) For $f \in C^\infty$; $\bigwedge_N(f)$ converges uniformly to $\bigwedge(f)$ together with any finite number of derivatives; therefore $T_u^p(x_0, (\bigwedge - \bigwedge_N)(f)) \to 0$ as $N \to \infty$.

III. **Applications to pointwise convergence.** Before we discuss the applications we shall introduce some of the notation to be used in this section.

Given a sequence $\{s_n\}_{n=0}^{\infty}$, define

$$s_n^{(0)} = s_n, \qquad s_n^{(j+1)} = \sum_{k=0}^{n} s_k^{(j)}$$

and

$$A_n^{(0)} = 1, \qquad A_n^{(j+1)} = \sum_{k=0}^{n} A_k^{(j)}.$$

$\sigma_n^{(k)} = s_n^{(k)}/A_n^{(k)}$ are the Cesàro means of $\{s_n\}$. If $\sigma_n^{(k)} \to s$ as $n \to \infty$ we shall say that s_n is summable (C, k). Finally we say that s_n is summable Abel if

$$\lim_{x \to 1^-} \sum_{n=0}^{\infty} s_n x^n$$

exists.

As an application of our results of §I we state

THEOREM 3. *Let $\{\lambda_n\}_{n=-\infty}^{\infty}$ be a sequence satisfying $|\lambda_n| \leqq C$;*

$$\sum_{\pm 2^k}^{\pm 2^{k+1}} |n|^r |\Delta^{r+1}(\lambda_n)| \leqq C \qquad (k = 0, 1, 2, \ldots)$$

and let $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ be a trigonometric series, summable (C, k) on a measurable set E.

Then the series $\sum_{n=-\infty}^{\infty} \lambda_n a_n e^{inx}$ is summable (C, k) a.e. in E, provided $0 \leqq k \leqq r-2$.

The proof of the theorem will be divided into three lemmas.

LEMMA 1 (M. WEISS). *If $\sum_{n=0}^{\infty} a_n e^{inx}$ is summable (C, k) on a measurable set E, then the $(k+1)$th termwise integrated series*

$$f(x) \sim \sum_{0}^{\infty} (in)^{-(k+1)} a_n e^{inx} \in t_{k+1}^{p}(x)$$

for almost every $x \in E$ $(1 \leqq p < \infty)$.

For the proof of Lemma 1 we refer the reader to [6, Theorem C′].

LEMMA 2. *If $g(x) = \sum_{n=-\infty}^{\infty} b_n e^{inx} \in t_{k+1}^{p}(x)$ for $x \in E$, then the $(k+1)$th termwise derivative of the series is*

(i) *summable $(C, k+1)$ a.e. in E,*

(ii) *summable $(C, k+3)$ everywhere in E.*

Proof. According to the corollary of Theorem 9 in [1], for any closed set $F \subset E$, we may decompose $g = g_1 + g_2$; $g_1, g_2 \in \mathscr{L}^p$; where

(a) g_1 has a classical $(k+1)$th $(\mathscr{L}^\infty)$ Peano derivative at every point of F,

(b) $((1/h) \int_{|x-x_0|<h} |g_2(x)|^p \, dx)^{1/p} = O(h^{k+1})$ and

$$\int_{-\pi}^{\pi} \frac{|g_2(x)|}{|x-x_0|^{k+2}} \, dx < \infty$$

(see [9, Theorem 10]).

If we now write the Fourier series expansion of g_1 and g_2 from property (a) it is known that the $(k+1)$th termwise derivative of the Fourier series of g_1 is $(k+1)$th Cesàro summable almost everywhere in F. (See [9, II, Theorem (5.4), p. 81].)

From property (b) and the fact that if $K_n^r(t)$ is the nth Cesàro kernel of the r means,

$$\left|\left(\frac{d}{dt}\right)^r K_n^r(t)\right| \leqq \frac{C}{|t|^{r+1}} \qquad (|t| \leqq \pi)$$

(see [9, II, p. 60]). The result follows for g_2 and hence for g, proving (i). The statement (ii) is a consequence of the fact that the primitive of $g(x)$ has a $(k+2)$th Peano derivative at every point of E and hence the $(k+2)$th termwise derivative of its Fourier series expansion is summable (C, α) for every $\alpha > k+2$. (See [9, II, Theorem (1.7), p. 60].)

The following lemma is due to A. Zygmund [8], [7].

LEMMA 3. *Let* $\{\lambda_n\}$ *be a sequence satisfying* $|\lambda_n| \leqq C$; $|\Delta^{k+1}(\lambda_n)| \leqq Cn^{-(k+1)}$. *Set* $s_n = \sum_{j=0}^n u_j$. *If* $s_n^{(k)} = o(n^k)$ *then for* $N = [1/(1-x)]$,

$$\sum_{n=0}^{\infty} \lambda_n u_n x^n - \sum_{n=0}^{N} s_n^{(k)} \Delta^{(k+1)}(\lambda_n) \to 0$$

as $x \to 1^-$.

Proof. A summation by parts argument shows that

$$\begin{aligned}
\sum_{n=0}^{\infty} \lambda_n u_n x^n &= \sum_{n=0}^{\infty} s_n^{(k)} \Delta^{(k+1)}(\lambda_n x^n) \\
&= \sum_{n=0}^{\infty} s_n^{(k)} \left\{ \sum_{j=0}^{k+1} \binom{k+1}{j} \Delta^j(x^n) \Delta^{k+1-j}(\lambda_{n+j}) \right\} \\
&= \sum_{n=0}^{\infty} s_n^{(k)} \Delta^{k+1}(\lambda_n) x^n + \sum_{j=1}^{k+1} \binom{k+1}{j} (1-x)^j \left\{ \sum_{n=0}^{\infty} s_n^{(k)} \Delta^{k+1-j}(\lambda_n) x^n \right\}.
\end{aligned}$$

Set $N = [1/(1-x)]$, then

$$\begin{aligned}
\sum_{n=0}^{\infty} \lambda_n u_n x^n &- \sum_{n=0}^{N} s_n^{(k)} \Delta^{k+1}(\lambda_n) \\
&= \sum_{n=0}^{N} s_n^{(k)} [\Delta^{k+1} \lambda_n](x^n - 1) + \sum_{n=N+1}^{\infty} s_n^{(k)} \Delta^{k+1}(\lambda_n) x^n \\
&\quad + \sum_{j=1}^{k+1} \binom{k+1}{j} (1-x)^j \left\{ \sum_{n=0}^{\infty} s_n^{(k)} \Delta^{k+1-j}(\lambda_n) x^n \right\} \\
&= \sum_{n=1}^{N} o(1)(1-x) + o\left(\frac{1}{N}\right) \sum_{n=N}^{\infty} x^n + \sum_{j=1}^{k+1} (1-x)^j o((1-x)^{-j}) = o(1),
\end{aligned}$$

since $|\Delta^j(\lambda_n)| \leqq Cn^{-j}$ for $1 \leqq j \leqq k+1$.

REMARK. Since $\sum_{n=0}^{\infty} s_n^{(k)} \Delta^{k+1}(\lambda_n)$ and $\sum_{n=0}^{\infty} u_n \lambda_n$ are equisummable (C, k), under the conditions of Lemma 3, we have that if the series $\sum \lambda_n u_n$ is summable Abel then it is also summable (C, k).

Proof of Theorem 3. We observe first (see [4] and [9, II, p. 216]) that if $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ is summable (C, k) in E, then the conjugate series $\sum_{n=-\infty}^{\infty} \operatorname{sgn}(n) a_n e^{inx}$ is summable (C, k) a.e. in E. Hence without loss of generality we may assume that $a_n = 0$ for $n < 0$.

Set $f(x) \sim \sum_{n=0}^{\infty} a_n (in)^{-(k+1)} e^{inx}$, using Lemma 1, $f(x) \in t_{k+1}^p(x)$ for almost every x in E $(1 \leqq p < \infty)$. Applying Theorem 2, $\bigwedge(f) \sim \sum_{n=0}^{\infty} \lambda_n (in)^{-(k+1)} a_n e^{inx}$ belongs to $t_{k+1}^p(x)$ a.e. in E.

From Lemma 2, $\sum_{n=0}^{\infty} \lambda_n a_n e^{inx}$ is summable $(C, k+1)$ a.e. in E and therefore summable Abel a.e. in E.

The theorem follows by applying the remark to Lemma 3.

As a consequence of Theorem 3 we obtain a recent result of M. Weiss and A. Zygmund when $\lambda_n = |n|^{i\gamma}$ (γ real), see [7], also [5].

Another interesting application is

THEOREM 4. *Let $F(t)$ be a bounded infinitely differentiable function on the real line and entire (real analytic) at infinity. Then $\lambda_n = F(n)$ is a multiplier sequence that preserves (C, k) summability almost everywhere, for every $k \geqq 0$. This is so because $|(d/dt)^n F(t)| \leqq C_n |t|^{-n}$. In particular $F(t)$ being the bounded ratio of two polynomials will satisfy the conditions of Theorem 4.*

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